JOURNAL OF APPROXIMATION THEORY 46, 213–216 (1986)

Limits of Some q-Laguerre Polynomials

RICHARD ASKEY*

Department of Mathematics, University of Wisconsin, Madison, Wisconsin 53706 U.S.A.

Communicated by Paul G. Nevai

Received August 7, 1984

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A q-analogue of Palama's limit, obtaining Hermite polynomials from Laguerre polynomials as the parameter $\alpha \to \infty$, is given, and the corresponding limit for a pair of weight functions is obtained. © 1986 Academic Press, Inc.

1. PALAMA'S LIMIT OF LAGUERRE POLYNOMIALS

Palama [5] proved that

$$\lim_{\alpha \to \infty} (2/\alpha)^{n/2} L_n^\alpha((2\alpha)^{1/2} x + \alpha) = (-1)^n H_n(x)/n!. \tag{1.1}$$

Here

$$L_n^\alpha(x) = \frac{(\alpha+1)_n}{n!}\, {}_1F_1\left(\begin{matrix} -n \\ \alpha+1 \end{matrix}; x\right),$$

$$H_n(x) = (2x)^n\, {}_2F_0\left(\begin{matrix} -n/2,\ (1-n)/2 \\ - \end{matrix}; \frac{-1}{x^2}\right),$$

$$(a)_n = \Gamma(n+a)/\Gamma(a),$$

and

$${}_pF_q\left(\begin{matrix} a_1,\dots, a_p \\ b_1,\dots, b_q \end{matrix}; x\right) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n x^n}{(b_1)_n \cdots (b_q)_n n!}.$$

* Supported in part by NSF Grant DMS-840071.

0021-9045/86 $3.00

A number of proofs have been given, but my favorite one has not appeared in a paper. The orthogonality relations for Laguerre and Hermite polynomials are

$$\int_0^\infty L_n^\alpha(x)\, L_m^\alpha(x)\, x^\alpha e^{-x}\, dx = 0, \qquad m \neq n, \tag{1.2}$$

$$\int_{-\infty}^\infty H_n(x)\, H_m(x)\, e^{-x^2}\, dx = 0, \qquad m \neq n. \tag{1.3}$$

To prove (1.1) it is sufficient to show that the weight function $w(x)$ in (1.2) can be changed so that it has $\exp(-x^2)$ as a limit, and to check that the coefficients of x^n on both sides of (1.1) are equal. The maximum of $x^\alpha e^{-x} = \exp(-x + \alpha \log x)$ occurs when $x = \alpha$, so shift $x = \alpha$ to $x = 0$. Then

$$\begin{aligned} w(x+\alpha) &= \exp(-x + \alpha \log(x+\alpha)) \exp(-\alpha) \\ &= \exp(-x + \alpha \log(1 + x/\alpha))\, w(\alpha) \\ &= \exp[-x + \alpha(x/\alpha - x^2/2\alpha^2 + O(x^3/\alpha^3))]\, w(\alpha) \\ &= \exp[-x^2/2\alpha^2 + O(x^3\alpha^{-2})]\, w(\alpha). \end{aligned}$$

This gives

$$w(x\sqrt{2\alpha} + \alpha) = \exp[-x^2 + O(x^3/\sqrt{\alpha})]\, w(\alpha).$$

The remaining factor in (1.1) is determined by matching up the coefficients of x^n.

I describe this proof as the third side of a commutative diagram with this arrow at infinity. For

$$\lim_{\beta \to \infty} P_n^{(\alpha,\beta)}\left(1 - \frac{2x}{\beta}\right) = L_n^\alpha(x)$$

and

$$\lim_{\beta \to \infty} \beta^{-n/2} P_n^{(\beta,\beta)}(x/\beta^{1/2}) = H_n(x)/2^n n!$$

are well known, and can be proven by the same argument as above. The limit (1.1) completes the diagram.

2. Some q-Laguerre Polynomials

There are a number of q-extensions of Laguerre and Hermite polynomials. In some cases one can just let $\alpha \to \infty$ in both the polynomials and weight functions. This happens for the continuous q-Laguerre polynomials in [1, p. 24] and the continuous q-Hermite polynomials of Rogers which also mentioned in [1, p. 24]. References for other treatments of Rogers' polynomials are given in [1, p. 24].

A more interesting case is a set of polynomials introduced by Hahn [3] and treated in some detail by Moak [4]. Define

$$(a; q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1}) \tag{2.1}$$

and

$$(a; q)_\infty = \prod_{k=0}^{\infty} (1-aq^k) \tag{2.2}$$

when $|q| < 1$. A set of q-Laguerre polynomials is given by

$$L_n^\alpha(x; q) = \frac{(q^{\alpha+1}; q)_n}{(q; q)_n} \sum_{k=0}^{n} \frac{(q^{-n}; q)_k \, q^{\binom{k}{2}} (q^{n+\alpha+1}x)^k}{(q^{\alpha+1}; q)_k (q; q)_k}. \tag{2.3}$$

One orthogonality is

$$\int_0^\infty L_n^\alpha(x; q)\, L_m^\alpha(x; q) \frac{x^\alpha \, dx}{(-x; q)_\infty} = 0, \qquad m \neq n. \tag{2.4}$$

Clearly

$$\lim_{\alpha \to \infty} L_n^\alpha(q^{-\alpha}x; q) = \frac{1}{(q; q)_n} \sum_{k=0}^{n} \frac{(q^{-n}; q)_k \, q^{\binom{k}{2}} (q^{n+1}x)^k}{(q; q)_k} = S_n(x; q). \tag{2.5}$$

These are polynomials considered by Stieltjes (in a special case) and Wigert [6], and shown to be orthogonal with respect to a log normal distribution. There are many other positive measures for which these polynomials and those defined in (2.3) are orthogonal, since the Stieltjes moment problems are indeterminate.

To find a measure which is the limit of the measure in (2.4) take α to be an integer k and consider

$$\frac{(q^{-k}x)^k}{(-q^{-k}x; q)_\infty} = \frac{q^{-k^2}x^k}{(-x; q)_\infty (-xq^{-k}; q)_k} = \frac{q^{(-k^2+k)/2}}{(-x; q)_\infty (-q/x; q)_k}.$$

Thus

$$\lim_{k \to \infty} \frac{q^{-(k^2+k)/2} x^k}{(-q^{-k}x; q)_\infty} = \frac{1}{(-x; q)_\infty (-q/x; q)_\infty}$$

and

$$\int_0^\infty \frac{S_n(x;q)\,S_m(x;q)\,dx}{(-x;q)_\infty(-q/x;q)_\infty}=0, \qquad m\neq n.$$

The value of

$$\int_0^\infty \frac{dx}{(-x;q)_\infty(-q/w;q)_\infty}$$

and of a more general q-beta integral

$$\int_0^\infty x^{c-1}\frac{(-xq^{a+c};q)_\infty(-q^{b+1-c}/x;q)_\infty}{(-x;q)_\infty(-q/x;q)_\infty}\,dx$$
$$=\frac{\Gamma(c)\,\Gamma(1-c)\,\Gamma_q(a)\,\Gamma_q(b)}{\Gamma_q(c)\,\Gamma_q(1-c)\,\Gamma_q(a+b)}$$

is given in [2]. Here $0<q<1$ and

$$\Gamma_q(x)=(q;q)_\infty(1-q)^{1-x}/(q^x;q)_\infty.$$

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